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Generalizations of Banach Contraction Mapping Principle

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Abstract *A large number of extensions of Banach Contraction Mapping Principle are attempted by many authors in many research papers. Rakotch used a decreasing function* ψ *on* $\mathbb{R}^+$ *to* $[0,\ 1)$ *for a contraction type condition and obtained a fixed point theorem. A slight variation of the Rakotch theorem is presented by Geraghty. In the theorem of Geraghty, the function* ψ *of Rakotch satisfies the condition that* $\psi(t_n) \to 1 \Rightarrow t_n \to 0$ *whereas in Rakotch it is a decreasing function* $\psi: \mathbb{R}^+ \to [0,\ 1)$. *Boyd and Wong obtained more general fixed point theorem by replacing the decreasing function in the theorem of Rakotch by an upper semi-continuous function. Matkowski in his fixed point theorem further modified the condition on the function* $\psi: \mathbb{R}^+ \to [0,\ 1)$ *of Rakotch by defining* $\psi: (0,\ \infty) \to (0,\ \infty)$ *to be monotone non-decreasing and satisfying the condition* $\lim_{n\to\infty} \psi^n(t) = 0$ *for all* $t > 0$. *Browder, Meer and Keeler, Kirk, Suzuki, Alber and Rhoades extended the results further. Three fixed point theorems are proved in this article by taking* ψ *to be an upper semi-continuous function from right. The function* ψ *is from the set of all positive real numbers to itself and appears out of the metric function as* $\psi(d(x,y))$. *Examples are provided to support the theorems. Finally the celebrated fixed point theorem by Kannan is generalized. In the next theorem an attempt has been made to take the function* ψ *inside the metric. Precisely,* ψ *is defined to be function from a general metric space to itself. Thus in the next theorem it appears like* $d(\psi(x), \psi(y))$. *This theorem is also illustrated by an example.*

Keywords: *Upper Semi-Continuous Function, Generalized Contractions, Fixed Point, T-Orbitally Complete Metric Space, Complete Metric Space.*

AMS Subject Classification: *47H10*

I. Introduction

Stefan Banach, a celebrated Mathematician from Poland, stated and proved the first astonishing fixed point theorem in 1922, known as the "Banach Contraction Mapping Principle" [2]. This theorem is the origin of Metric Fixed Point Theory. Fixed points, Banach Contraction Mapping Principle and Brower Fixed Point theorem are widely used in many branches of Mathematics. See some of them in [16]. Especially non-linear differential equations can be solved by using Banach Contraction Mapping Principle. Later on this principle has been generalized by many Mathematicians in many different ways. See Alber [1], Baillon [5], Kirk [6], Meer and Keeler [9], Rhoades [13], and Suzuki [15]. In fact vast literature is available regarding the generalization and extension of the noteworthy principle. In this research paper some generalizations of Banach Contraction Mapping Principle are proved. In these results we have modified the conditions of the Banach Contraction Mapping Principle and obtained three fixed point theorems. Kannan invented new type of contractions called Kannan Mappings. Kannan proved that his contractions are independent of Banach contractions and also proved that every Kannan mapping on a complete metric space has a unique fixed point [17]. Lj. B. Ćirić in [18] introduced Generalized Contractions and also proved Generalized Contractions include Banach Contractions and Kannan Contractions. Lj. B. Ćirić in the same research paper, proved a fixed point theorem which is a generalization of both Kannan and Banach Fixed Point Theorems. Lj. B. Ćirić diluted the condition of completeness of a metric space to T-Orbitally Completeness. In this research paper a fixed point theorem by using a somewhat light mode of Generalized Contractions of Lj. B. Ćirić is proved. This theorem can be illustrated by exhibiting examples from the metric spaces like $\mathbb{R}^+$ and l^2. For instance, an example of the metric space $X = [0,10]$ with the absolute value metric is given.

II. Preliminaries and Definitions

Definition 3.1 (Metric Space) [7]: - A "Metric Space" is a pair $(X,\ d)$, where X is a set and d is a metric on X (or distance function on X), that is, a function defined on $X \times X$ such that for all $x,\ y,\ z \in X$ we have:

(M1) d is real-valued, finite and non-negative.

(M2) $d(x,\ y) = 0$ if and only if $x = y$.

(M3) $d(x,\ y) = d(y,\ x)$ (Symmetry).

(M4) $d(x,\ y) \le d(x,\ z) + d(z,\ y)$ (Triangle Inequality).

Example 3.1 [7]: - The set of all real numbers, taken with the usual metric defined by $d(x, y) = |x - y|$ is a metric space.
Note 3.1 [12]: - It is important to note that if (X, d) is a metric space and $A \subseteq X$, then (A, d) is also a metric space.
Definition 3.2 [7]: - A “Fixed Point” of a mapping $T: X \to X$ of a set X into itself is an $x \in X$ which is mapped onto itself, that is $Tx = x$.
Definition 3.3 [14]: - Let (X, d) be a metric space and let T be a mapping on$X$. Then T is called a “Contraction” if there exists $r \in [0,1)$such that $d(Tx, Ty) \leq rd(x, y)\, \forall x, y \in X$
The following famous theorem is referred to as the Banach Contraction Mapping Principle.
Theorem 3.1 (Banach) [2]: - Let (X, d) be a complete metric space and let T be a contraction on X . Then T has a unique fixed point.
Definition 3.4: - A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space (X, d) is said to converge or to be convergent if there is an $x \in X$ such that$\lim_{n\to\infty} d(x_n, x) = 0$, x is called the limit of $\{x_n\}_{n=1}^{\infty}$ and we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
Definition 3.5: - A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space (X, d) is said to be a “Cauchy Sequence” is for every $\varepsilon > 0$ there is an $N = N(\varepsilon)$ such that $d(x_m, x_n) < \varepsilon$ for every $m, n \geq N$.
Theorem 3.2: - Every convergent sequence in a metric space is a Cauchy sequence.
Note 3.2: - The converse of the above theorem is not true in general. That is a Cauchy sequence in a metric space X may or may not converge in$X$.
Definition 3.6: - A metric space (X, d) is said to be a “Complete Metric Space” if every Cauchy Sequence in X converges in X.
Definition 3.7: - A function $\psi: \mathbb{R} \to [0, \infty)$ is said to be an “Upper Semi-Continuous from right” if for any sequence $\{t_n\}_{n=1}^{\infty}$ converging to $t \geq 0$, $\limsup_{n\to\infty} \psi(t_n) \leq \psi(t)$.
Example 3.2: - Define the function $\psi: \mathbb{R}^+ \to [0, \infty)$ as follows:

$$\psi(t) = \begin{cases} \sqrt{t}, if\ t \in [0, 1), \\ \sqrt{t} + 1, if\ t \in [1, \infty). \end{cases}$$

The function is discontinuous at $t = 1$. We have $\psi(1) = 2$. As $t \to 1$ from right $\lim_{t\to 1} \psi(t) = 2 \leq 2$ and from left $\limsup_{t\to 1} \psi(t) = 1 \leq 2$. Thus the function is upper semi-continuous from right.
Theorem 3.3 (Rakotch) [11]: - Let (X, d) be a complete metric space and suppose that $T: X \to X$ satisfies $d(Tx, Ty) \leq \psi(d(x, y))d(x, y)$ for all $x, y \in X$, where ψ is a decreasing function on $\mathbb{R}^+$ to $[0, 1)$. Then T has a unique fixed point.
A slight variation of the Rakotch theorem 3.3 is given by Geraghty as follows.
Theorem 3.4 (Geraghty) [4]: - Let (X, d) be a complete metric space and suppose that $T: X \to X$ satisfies$d(Tx, Ty) \leq \psi(d(x, y))d(x, y)$ for all $x, y \in X$where $\psi(t_n) \to 1 \Rightarrow t_n \to 0$. Then T has a unique fixed point.
Theorem 3.5 (Boyd-Wong) [3]: - Let (X, d) be a complete metric space and suppose that $T: X \to X$ satisfies $d(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$, where, $\psi: \mathbb{R} \to [0, \infty)$is upper semi continuous from the right and satisfies $0 \leq \psi(t) < t$ for all $t > 0$. Then T has a unique fixed point in$X$.
Matkowski replaced the condition of upper semi-continuity on ψ by the condition and stated and proved the following theorem.
Theorem 3.6 (Matkowski) [8]: - Let (X, d) be a complete metric space and suppose that $T: X \to X$ satisfies $d(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$, where, $\psi: (0, \infty) \to (0, \infty)$is monotone non-decreasing and satisfies $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$. Then T has a unique fixed point in X.
Meer and Keeler used a diverse approach and extended the theorem of Boyd and Wong as follows.
Theorem 3.7 (Meer and Keeler) [9]: - Let (X, d) be a complete metric space and suppose that $T: X \to X$ satisfies the condition: for each$\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$, $\varepsilon \leq d(x, y) \leq \varepsilon + \delta \Rightarrow d(Tx, Ty) \leq \varepsilon$. Then T has a unique fixed point.
Rhoades extended the Banach Contraction Mapping Principle as follows.
Theorem 3.8 (Rhoades) [13]: - Let (X, d) be a complete metric space and suppose that $T: X \to X$ satisfies $d(Tx, Ty) \leq d(x, y) - \psi(d(x, y))$ for all $x, y \in$, where $\psi: (0, \infty) \to (0, \infty)$ is continuous and non-decreasing function such that $\psi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.
Definition 3.5: Let T maps a metric space X into X. Then the metric space X is said to “T-Orbitally Complete” if every Cauchy sequence of the form $\{T^n x: x \in X\}_{n=1}^{\infty}$ has a limit in X.
Definition 3.9: Let (X, d) be a metric space and let Tbe a mapping on X. Then T is called “Kannan Mapping” if there exists $r \in [0,1/2)$such that $d(Tx, Ty) \leq rd(x, Tx) + rd(y, Ty)$ for all $x, y \in X$.

Theorem 3.9 (Kannan) [16]: Let (X, d) be a complete metric space and let $T: X \to X$ be satisfy the condition $d(Tx, Ty) \le r\{d(x, Tx) + d(y, Ty)\}$, for all $x, y \in X$, where $r \in \left[0, \frac{1}{2}\right)$, then T has unique fixed point in X.
Lj. B. Ćirić introduced the following notion of Generalized Contractions and proved the subsequent theorem.
Definition 3.10 [17]: Let (X, d) be Metric Space and let T be a mapping of Xinto itself. Then T is said to be a λ Generalized Contraction if for every $x, y \in X$ there exists non-negative numbers $p(x, y)$, $q(x, y)$, $r(x, y)$, $s(x, y)$ such that $\underset{x,y\in X}{Sup}\{p(x, y) + q(x, y) + r(x, y) + 2s(x, y)\} = \lambda < 1$ and
$d(Tx, Ty) \le pd(x, y) + qd(x, Tx) + rd(y, Ty) + s\{d(x, Ty) + d(y, Tx)\}$ for all $x, y \in X$.
Theorem 3.10 [17]: Let T be $\lambda -$generalized contraction of T-Orbitally Complete metric space X into itself. Then
1) There is a unique point in $u \in X$which is a fixed point under T,
2) $T^n x \to u$ for every $x \in X$,
3) $d(T^n x, u) \le \frac{\lambda^n}{1-\lambda} d(x, Tx)$

III. Main Results

Now some fixed point theorems are proved in which the function $\psi: \mathbb{R} \to [0, \infty)$ which is an upper semi-continuous from right is used.
Theorem 4.1: - Let (X, d) be a complete metric space and suppose that $T: X \to X$ satisfies $d(Tx, Ty) \le \psi(\alpha d(x, Tx) + \beta d(y, Ty))$ for all $x, y \in X$, where, $\psi: \mathbb{R} \to [0, \infty)$is upper semi continuous from the right and satisfies $0 \le \psi(t) < t$ for all$t > 0$, $\psi(0) =$. Also $0 < \alpha + \beta < 1$, $\alpha > 0$, $\beta > 0$. Then T has a unique fixed point in$X$.
Proof: - Let $x_0 \in X$ be an arbitrary but a fixed element in X. Define a sequence of iterates $\{x_n\}_{n=1}^{\infty}$ in X by $x_1 = Tx_0$, $x_2 = Tx_1 = T^2 x_0$, $x_3 = Tx_2 = T^3 x_0, \ldots\ldots, x_n = Tx_{n-1} = T^n x_0, \ldots\ldots$ Now consider,

$$d(x_{n+1}, x_n) < d(Tx_n, Tx_{n-1})$$
$$\le \psi(\alpha d(x_n, Tx_n) + \beta d(x_{n-1}, Tx_{n-1}))$$
$$= \psi(\alpha d(x_n, x_{n+1}) + \beta d(x_{n-1}, Tx_n))$$
$$< \alpha d(x_n, x_{n+1}) + \beta d(x_{n-1}, Tx_n) (\because \psi(t) < t)$$

Thus

$$d(x_{n+1}, x_n) < \alpha d(x_n, x_{n+1}) + \beta d(x_{n-1}, x_n)$$
$$\therefore (1-\alpha) d(x_{n+1}, x_n) < \beta d(x_{n-1}, x_n)$$
$$\therefore d(x_{n+1}, x_n) < \frac{\beta}{1-\alpha} d(x_{n-1}, x_n)$$
$$\therefore d(x_{n+1}, x_n) < h d(x_{n-1}, x_n).$$

Where, $h = \frac{\beta}{1-\alpha}$. Here $0 < h < 1$ because $0 < \alpha + \beta < 1$, $\alpha > 0$, $\beta > 0$. Continue in this way to get $d(x_{n+1}, x_n) < h^n d(x_0, x_1)$. Taking limit as $n \to \infty$, $d(x_{n+1}, x_n) \to 0$ $(\because 0 < h < 1)$. Therefore $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. As X is a complete metric space, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. It is shown below that x is a fixed point of T. As T a is continuous function, $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = T\left(\lim_{n\to\infty} x_{n-1}\right) = Tx$. Therefore $Tx = x$ and x is a fixed point of T. Next to show that x is unique fixed point of T. Let $y \in X$ be another fixed point of T. Consider

$$d(x, y) = d(Tx, Ty) \le \psi(\alpha d(x, Tx) + \beta d(y, Ty))$$
$$= \psi(\alpha d(x, x) + \beta d(y, y)) \qquad (\because Tx = x, Ty = y)$$
$$= \psi(0) \qquad (See\ M(2)\ of\ definition\ 3.1)$$
$$= 0 \qquad (\because \psi(0) = 0)$$

$$\therefore d(x, y) = 0$$

Thus$x = y$, by M(2) of definition 3.1, and hence the fixed point of T is unique.
Theorem 4.2: - Let (X, d) be a complete metric space and suppose that $T: X \to X$ satisfies $d(Tx, Ty) \le \alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty))$ for all $x, y \in X$, where, $\psi: \mathbb{R} \to [0, \infty)$ is upper semi continuous from the right and satisfies $0 \le \psi(t) < t$ for all$t > 0$, $\psi(0) = 0$. Also $0 < \alpha + \beta < 1$, $\alpha > 0$, $\beta > 0$. Then T has a unique fixed point in$X$.
Proof: - Let $x_0 \in X$ be an arbitrary but a fixed element in X. Define a sequence of iterates $\{x_n\}_{n=1}^{\infty}$ in X by $x_1 = Tx_0$, $x_2 = Tx_1 = T^2 x_0$, $x_3 = Tx_2 = T^3 x_0, \ldots\ldots, x_n = Tx_{n-1} = T^n x_0, \ldots\ldots$ Now consider,

$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1})$
$\leq \alpha\psi(d(x_n, Tx_n)) + \beta\psi(d(x_{n-1}, Tx_{n-1}))$
$= \alpha\psi(d(x_n, x_{n+1})) + \beta\psi(d(x_{n-1}, x_n))$
$< \alpha d(x_n, x_{n+1}) + \beta d(x_{n-1}, x_n)(\because \psi(t) < t)$
Thus,

$d(x_{n+1}, x_n) < \alpha d(x_n, x_{n+1}) + \beta d(x_{n-1}, x_n)$
$\therefore (1-\alpha)d(x_{n+1}, x_n) < \beta d(x_{n-1}, x_n)$
$\therefore d(x_{n+1}, x_n) < \frac{\beta}{1-\alpha} d(x_{n-1}, x_n)$
$\therefore d(x_{n+1}, x_n) < hd(x_{n-1}, x_n).$

Where, $h = \frac{\beta}{1-\alpha}$. Here $0 < h < 1$ because $0 < \alpha + \beta < 1, \alpha > 0, \beta > 0$. Continue to get $d(x_{n+1}, x_n) < h^n d(x_0, x_1)$. Taking limit as $n \to \infty$, gives $d(x_{n+1}, x_n) \to 0$. Therefore $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. As X is a complete metric space, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. We shall show that x is a fixed point of T. As T a is continuous function we have, $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = T\left(\lim_{n\to\infty} x_{n-1}\right) = Tx$. Therefore $Tx = x$ and x is a fixed point of T. Next we shall show that x is unique fixed point of T. Let $y \in X$ be another fixed point of T. Consider

$d(x, y) = d(Tx, Ty) \leq \alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty))$
$= \alpha\psi(d(x, x)) + \beta\psi(d(y, y)) \quad (\because Tx = x, Ty = y)$
$= \alpha\psi(0) + \beta\psi(0) \quad (See\ M(2)\ of\ definition\ 3.1)$
$= 0 \quad (\because \psi(0) = 0)$
$\therefore d(x, y) = 0$

Thus $x = y$ and hence the fixed point of T is unique.

Example 4.1: - Consider the metric space $(\mathbb{R}^+, |\ \ |)$, that is the metric space of non-negative real numbers with the absolute value metric. See example 3.1 and note 3.1. This metric space is a complete metric space. Define the function $T: \mathbb{R}^+ \to \mathbb{R}^+$ by $Tx = \frac{x}{13}$. Define the function $\psi: \mathbb{R}^+ \to [0, \infty)$by

$$\psi(t) = \begin{cases} \frac{t}{4}, & if\ 0 \leq t < 2, \\ \frac{t}{2}, & if\ 2 \leq t < \infty \end{cases}$$

It is straight forward to see that the function $\psi(t)$defined above is continuous at every point except at $t = 2$. As $t \to 2$ from right we have $\lim_{t\to 2} \psi(t) = 1 \leq 1$and from left we have $\limsup_{t\to 2} \psi(t) = \frac{1}{2} \leq 1$, where $\psi(2) = 1$. Thus clearly the function $\psi(t)$ is upper semi-continuous from right. It also satisfies $0 < \psi(t) < t$ for $t > 0$, because the $\frac{t}{4} < t, \frac{t}{2} < t\ \forall t \in \mathbb{R}^+$. Also $\psi(0) = 0$. Choose $\alpha = \beta = \frac{1}{3}$. Then we can verify that the condition in the theorem 4.1, that is $d(Tx, Ty) \leq \psi(\alpha d(x, Tx) + \beta d(y, Ty))$ for all $x, y \in \mathbb{R}^+$ is satisfied. We observe that $d(Tx, Ty) = d\left(\frac{x}{13}, \frac{y}{13}\right) = \frac{|x-y|}{13}$. Also $\psi(\alpha d(x, Tx) + \beta d(y, Ty)) = \psi\left(\frac{1}{3} d\left(x, \frac{x}{13}\right) + \frac{1}{3} d\left(y, \frac{y}{13}\right)\right) = \psi\left(\frac{1}{3}\left(\frac{12x}{13}\right) + \frac{1}{3}\left(\frac{12y}{13}\right)\right) = \psi\left(\frac{4(x+y)}{13}\right)$. Now we consider all the three cases of values of $\frac{4(x+y)}{13}$.

Case1. $\frac{4(x+y)}{13} = 0$.

Case 2. $0 < \frac{4(x+y)}{13} < 2$.

Case 3. $2 \leq \frac{4(x+y)}{13} < \infty$.

Case 1. $\frac{4(x+y)}{13} = 0$. That is $x = 0, y = 0$, because $x \geq 0, y \geq 0$. Then $d(Tx, Ty) = \frac{|x-y|}{13} = \frac{|0-0|}{13} = \frac{0}{13} = 0$. And $\psi\left(\frac{4(x+y)}{13}\right) = \psi(0) = 0$. Thus $d(Tx, Ty) = 0 \leq 0 = \psi(\alpha d(x, Tx) + \beta d(y, Ty))$.

Case 2. $0 < \frac{4(x+y)}{13} < 2$. That is $0 < x + y < \frac{26}{4} = \frac{13}{2}$. Then $d(Tx, Ty) = \frac{|x-y|}{13}$. And $\psi\left(\frac{4(x+y)}{13}\right) = \frac{\frac{4(x+y)}{13}}{4} = \frac{x+y}{13}$. Clearly $\frac{|x-y|}{13} \leq \frac{x+y}{13}$ for all x, y satisfying $0 < x + y < \frac{26}{4} = \frac{13}{2}$. Hence $d(Tx, Ty) = \frac{|x-y|}{13} \leq \frac{x+y}{13} = \psi(\alpha d(x, Tx) + \beta d(y, Ty))$.

Case 3. $2 \leq \frac{4(x+y)}{13} < \infty$. That is $\frac{13}{2} = \frac{26}{4} \leq x + y < \infty$. Then $d(Tx, Ty) = \frac{|x-y|}{13}$. $\psi\left(\frac{4(x+y)}{13}\right) = \frac{\frac{4(x+y)}{13}}{2} = \frac{2(x+y)}{13}$. Clearly $\frac{|x-y|}{13} \leq \frac{2(x+y)}{13}$ for all x, y satisfying $\frac{26}{4} \leq x + y < \infty$. Hence $d(Tx, Ty) = \frac{|x-y|}{13} \leq \frac{2(x+y)}{13} = \psi(\alpha d(x, Tx) + \beta d(y, Ty))$.

Thus in all the cases we have $d(Tx, Ty) \leq \psi(\alpha d(x, Tx) + \beta d(y, Ty))$.

Therefore the condition of the theorem 4.1 is satisfied. We observe that $x = 0$ is the unique fixed point of T.
Example 4.2: - Consider the metric space $(\mathbb{R}^+, |\quad|)$, that is the metric space of non-negative real numbers with the absolute value metric. This metric space is a complete metric space. Define the function $T: \mathbb{R}^+ \to \mathbb{R}^+$and the function $\psi: \mathbb{R}^+ \to [0, \infty)$ as in example 4.1 above. Choose $\alpha = \beta = \frac{1}{3}$. Then we can verify that the condition in the theorem 4.2, that is $d(Tx, Ty) \le \alpha\psi\big(d(x, Tx)\big) + \beta\psi\big(d(y, Ty)\big)$ for all $x, y \in \mathbb{R}^+$is satisfied. We observe that $d(Tx, Ty) = d\left(\frac{x}{13}, \frac{y}{13}\right) = \frac{|x-y|}{13}$. Also $\alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty)) = \alpha\psi\left(d\left(x, \frac{x}{13}\right)\right) +$
$\beta\psi\left(d\left(y, \frac{y}{13}\right)\right) = \frac{1}{3}\psi\left(\frac{12x}{13}\right) + \frac{1}{3}\psi\left(\frac{12y}{13}\right)$.
Now we consider the all the four cases of values of $\frac{12x}{13}$and $\frac{12y}{13}$.
Case 1. $\frac{12x}{13} = 0, \frac{12y}{13} = 0$.
Case 2. $\frac{12x}{13} = 0, \frac{12y}{13} \ne 0$.
Sub-case 1. $0 < \frac{12y}{13} < 2$.
Sub-case 2. $2 \le \frac{12y}{13} < \infty$.
Case 3. $\frac{12x}{13} \ne 0, \frac{12y}{13} = 0$.
Sub-case 1. $0 < \frac{12x}{13} < 2$.
Sub-case 2. $2 \le \frac{12x}{13} < \infty$.
Case 4. $\frac{12x}{13} \ne 0, \frac{12y}{13} \ne 0$.
Sub-case 1. $0 < \frac{12x}{13} < 2,\ 0 < \frac{12y}{13} < 2$.
Sub-case 2. $0 < \frac{12x}{13} < 2,\ 2 \le \frac{12y}{13} < \infty$.
Sub-case 3. $2 \le \frac{12x}{13} < \infty,\ 0 < \frac{12y}{13} < 2$.
Sub-case 4. $2 \le \frac{12x}{13} < \infty,\ 2 \le \frac{12y}{13} < \infty$.
Case 1: -$\frac{12x}{13} = \frac{12y}{13} = 0$, that is $x = y = 0$. Then $d(Tx, Ty) = \frac{|x-y|}{13} = \frac{|0-0|}{13} = \frac{0}{13} = 0$.And $\alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty)) = \frac{1}{3}\psi(0) + \frac{1}{3}\psi(0) = 0 + 0 = 0$. Thus $d(Tx, Ty) = 0 \le 0 = \alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty))$.
Case 2: -$\frac{12x}{13} = 0, \frac{12y}{13} \ne 0$. That is $x = 0$ and $y \ne 0$.
Sub-case 1 of case 2: - $0 < \frac{12y}{13} < 2$. That is $0 < y < \frac{26}{12} = \frac{13}{6}$. Then $d(Tx, Ty) = \frac{|x-y|}{13} = \frac{|0-y|}{13} = \frac{y}{13}$.
And $\alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty)) = \frac{1}{3}\psi(0) + \frac{1}{3}\psi\left(\frac{12y}{13}\right) = 0 + \frac{1}{3}\frac{\left(\frac{12y}{13}\right)}{4} = \frac{y}{13}$. Thus $d(Tx, Ty) = \frac{y}{13} \le \frac{y}{13} = \alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty))$.
Sub-case 2 of case 2: - $2 \le \frac{12y}{13}$. That is $\frac{26}{12} = \frac{13}{6} \le y < \infty$. Then $d(Tx, Ty) = \frac{|x-y|}{13} = \frac{|0-y|}{13} = \frac{y}{13}$.
And $\alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty)) = \frac{1}{3}\psi(0) + \frac{1}{3}\psi\left(\frac{12y}{13}\right) = 0 + \frac{1}{3}\frac{\left(\frac{12y}{13}\right)}{2} = \frac{2y}{13}$. Thus $d(Tx, Ty) = \frac{y}{13} \le \frac{2y}{13} = \alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty))$.
Case 3: - $\frac{12x}{13} \ne 0, \frac{12y}{13} = 0$. That is $x \ne 0$ and $y = 0$. This case is similar to the case 2, with x and y interchanged.
Case 4: -$\frac{12x}{13} \ne 0, \frac{12y}{13} \ne 0$. That is $x \ne 0$and $y \ne 0$.
Sub-case 1 of case 4: - $0 < \frac{12x}{13} < 2, 0 < \frac{12y}{13} < 2$. Then $d(Tx, Ty) = \frac{|x-y|}{13}$. And $\alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty)) = \frac{1}{3}\psi\left(\frac{12x}{13}\right) + \frac{1}{3}\psi\left(\frac{12y}{13}\right) = \frac{1}{3}\frac{\left(\frac{12x}{13}\right)}{4} + \frac{1}{3}\frac{\left(\frac{12y}{13}\right)}{4} = \frac{x+y}{13}$. Thus $d(Tx, Ty) = \frac{|x-y|}{13} \le \frac{x+y}{13} = \alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty))$.
Sub-case 2 of case 4: - $0 < \frac{12x}{13} < 2, 2 \le \frac{12y}{13} < \infty$. Then $d(Tx, Ty) = \frac{|x-y|}{13}$. And $\alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty)) = \frac{1}{3}\psi\left(\frac{12x}{13}\right) + \frac{1}{3}\psi\left(\frac{12y}{13}\right) = \frac{1}{3}\frac{\left(\frac{12x}{13}\right)}{4} + \frac{1}{3}\frac{\left(\frac{12y}{13}\right)}{2} = \frac{x+2y}{13}$.
$d(Tx, Ty) = \frac{|x-y|}{13} \le \frac{x+2y}{13} = \alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty))$.
Sub-case 3 of case 4: - $2 \le \frac{12x}{13} < \infty, 0 < \frac{12y}{13} < 2$. This case is similar to the sub-case 2 above, with x and y interchanged and we conclude that $d(Tx, Ty) = \frac{|x-y|}{13} \le \frac{2x+y}{13} = \alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty))$

Sub-case 4 of case 4: - $2 \le \frac{12x}{13} < \infty$, $2 \le \frac{12y}{13} < \infty$. Then $d(Tx, Ty) = \frac{|x-y|}{13}$. And $\alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty)) = \frac{1}{3}\psi\left(\frac{12x}{13}\right) + \frac{1}{3}\psi\left(\frac{12y}{13}\right) = \frac{1}{3}\frac{\left(\frac{12x}{13}\right)}{2} + \frac{1}{3}\frac{\left(\frac{12y}{13}\right)}{2} = \frac{2(x+y)}{13}$. Thus $d(Tx, Ty) = \frac{|x-y|}{13} \le \frac{2(x+y)}{13} = \alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty))$.

Thus in all cases $d(Tx, Ty) \le \alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty))$. Thus the condition of the theorem 4.2 is satisfied. We observe that $x = 0$ is the unique fixed point of T.

Remark 4.1: - If we take ψ to be an identity mapping and $\alpha = \beta = \frac{1}{3}$ in the theorems 4.1 and 4.2, then we get the Kannan Fixed Point Theorem [10] for $r = \frac{1}{3}$, which states as follows: Let (X, d) be a complete metric space and let $T: X \to X$ be satisfy the condition $d(Tx, Ty) \le r\{d(x, Tx) + d(y, Ty)\}$, $\forall\, x, y \in X$, where $r \in \left[0, \frac{1}{2}\right)$, then T has unique fixed point in X.

Theorem 4.3: Let (X, d) be a complete metric space and suppose that $T: X \to X$ satisfies

(A) $d(Tx, Ty) \le \alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty)) + \gamma\psi(d(x, y))$ for all $x, y \in X$ where, $\psi: \mathbb{R} \to [0, \infty)$ is upper semi-continuous from the right and satisfies $0 \le \psi(t) < t$ for all $t > 0$, $\psi(0) = 0$. Also $0 < \alpha + \beta + \gamma < 1$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$. Then T has a unique fixed point in X.

Proof: Let $x_0 \in X$ be an arbitrary but a fixed element in X. Define a sequence of iterates $\{x_n\}_{n=1}^{\infty}$ in X by $x_1 = Tx_0$, $x_2 = Tx_1 = T^2x_0$, $x_3 = Tx_2 = T^3x_0, \ldots\ldots, x_n = Tx_{n-1} = T^nx_0, \ldots\ldots$

By the condition (A) on T we get,

$$\begin{gathered} d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \\ \le \alpha\psi(d(x_{n-1}, Tx_{n-1})) + \beta\psi(d(x_n, Tx_n)) + \gamma\psi(d(x_{n-1}, x_n)) \\ = \alpha\psi(d(x_{n-1}, x_n)) + \beta\psi(d(x_n, x_{n+1})) + \gamma\psi(d(x_{n-1}, x_n)) \\ < \alpha d(x_{n-1}, x_n) + \beta d(x_n, x_{n+1}) + \gamma d(x_{n-1}, x_n) (\because \psi(t) < t) \\ Thus, \\ d(x_n, x_{n+1}) < \alpha d(x_{n-1}, x_n) + \beta d(x_n, x_{n+1}) + \gamma d(x_{n-1}, x_n) \\ \therefore (1-\beta) d(x_{n+1}, x_n) < (\alpha + \gamma) d(x_{n-1}, x_n) \\ \therefore d(x_{n+1}, x_n) < \frac{\alpha + \gamma}{1 - \beta} d(x_{n-1}, x_n) \\ \therefore d(x_{n+1}, x_n) < h d(x_{n-1}, x_n) \end{gathered}$$

Where, $h = \frac{\alpha+\gamma}{1-\beta}$. Here $0 < h < 1$ because $0 < \alpha + \beta + \gamma < 1$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$. Continue in this way to get $d(x_{n+1}, x_n) < h^n d(x_0, x_1)$. Taking limit as $n \to \infty$ gives, $d(x_{n+1}, x_n) \to 0$ $(\because 0 < h < 1)$. Therefore $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. As X is a complete metric space, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. It is now shown that x is a fixed point of T. As T a is continuous function, $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = T\left(\lim_{n\to\infty} x_{n-1}\right) = Tx$. Therefore $Tx = x$ and x is a fixed point of T. Next to show that x is unique fixed point of T. Let $y \in X$ be another fixed point of T. Again by the condition (A),

$$\begin{aligned} d(x, y) = d(Tx, Ty) &\le \alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty)) + \gamma\psi(d(x, y)) \\ &= \alpha\psi(d(x, x)) + \beta\psi(d(y, y)) + \gamma\psi(d(x, y)) \\ &= \alpha\psi(0) + \beta\psi(0) + \gamma\psi(d(x, y)) \\ &= \gamma\psi(d(x, y)) \\ &< \gamma d(x, y) \end{aligned}$$

$$\text{Thus, } d(x, y) < \gamma d(x, y)$$

This is possible if and only if $d(x, y) = 0$, because $\gamma < 1$. Thus $x = y$ and hence the fixed point of $T$is unique.

Example 4.3: Consider the complete metric space of all non-negative real numbers with absolute value metric. Suppose that $T: X \to X$ defined by $Tx = \frac{x}{8}$. Let, $\psi: \mathbb{R} \to [0, \infty)$ is defined by $\psi(t) = \frac{t}{2}$. The function $\psi(t)$ is continuous (and hence upper semi-continuous from right), also $0 < \psi(t) < t$ for all $t > 0$, $\psi(0) = 0$. Let $\alpha = \beta = \gamma = \frac{1}{4}$. Then clearly $0 < \alpha + \beta + \gamma = \frac{3}{4} < 1$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$. Observe that $d(Tx, Ty) = d\left(\frac{x}{8}, \frac{y}{8}\right) = \frac{|x-y|}{8}$. Also

$$\alpha d(x,Tx)+\beta d(y,Ty)+\gamma d(x,y)$$
$$=\frac{1}{4}\psi\left(d\left(x,\frac{x}{8}\right)\right)+\frac{1}{4}\psi\left(d\left(y,\frac{y}{8}\right)\right)+\frac{1}{4}\psi(d(x,y))$$
$$=\frac{1}{4}\psi\left(\frac{7x}{8}\right)+\frac{1}{4}\psi\left(\frac{7y}{8}\right)+\frac{1}{4}\psi(|x-y|)$$
$$=\frac{1}{4}\frac{\left(\frac{7x}{8}\right)}{2}+\frac{1}{4}\frac{\left(\frac{7y}{8}\right)}{2}+\frac{1}{4}\frac{|x-y|}{2}=\frac{7x}{64}+\frac{7y}{64}+\frac{|x-y|}{8}$$
$$=\frac{7(x+y)}{64}+\frac{|x-y|}{8}.$$

Thus clearly $d(Tx,Ty)=d\left(\frac{x}{8},\frac{y}{8}\right)=\frac{|x-y|}{8}<\frac{7(x+y)}{64}+\frac{|x-y|}{8}=\alpha d(x,Tx)+\beta d(y,Ty)+\gamma d(x,y)$for all $x\in\mathbb{R}^+$.
The condition (A) of the theorem 4.3 is satisfied. $x=0$ is the unique fixed point of the function T.

Theorem 4.4: Let (X,d) be a T-Orbitally Complete metric space with respect to a mapping $T:X\to X$. Let $*$ be the operation defined on the set X by $\alpha * x$ for $\alpha\in\mathbb{R}$ and $x\in X$, such that $\alpha * x\in X$. Also let the metric d satisfies $d(\alpha x,\alpha y)=|\alpha|d(x,y)$ for any scalar $\alpha\in\mathbb{R}$. Further let there exist a number $k\in\mathbb{R}^+$ such that,

(B) $\underset{x,y\in X}{Sup}\,(p(x,\,y)+q(x,\,y)+r(x,\,y)+2s(x,\,y))=\lambda<\frac{1}{k}$ for some non-negative numbers $p(x,\,y)$, $q(x,\,y)$, $r(x,\,y)$, $s(x,\,y)$, which may depend on x and y,

(C) For the function $\psi:X\to X$ defined by $\psi(x)=kx,\,k\in\mathbb{R}$, T satisfies the condition $d(Tx,Ty)\le pd(\psi(x),\psi(y))+qd(\psi(x),\psi(Tx))+rd(\psi(y),\psi(Ty))+s\{d(\psi(x),\psi(Ty))+d(\psi(y),\psi(Tx))\}$ for all $x,\,y\in X$.

Then T has a unique fixed point in X.

Proof: For the sake of simplicity, denote $\alpha * x$ by αx. Let $x_0\in X$ be arbitrary and define a sequence, $x_1=Tx_0,\,x_2=Tx_1,\,x_3=Tx_2,\,\ldots\ldots,\,x_{n+1}=Tx_n,\,\ldots\ldots$. Now by using the condition (C) in the statement of the theorem and triangle inequality, gives

$$d(x_n,x_{n+1})$$
$$=d(Tx_{n-1},Tx_n)$$
$$\le pd(\psi(x_{n-1}),\psi(x_n))+qd(\psi(x_{n-1}),\psi(Tx_{n-1}))+rd(\psi(x_n),\psi(Tx_n))+s\{d(\psi(x_{n-1}),\psi(Tx_n))+d(\psi(x_n),\psi(Tx_{n-1}))\}$$
$$=pd(\psi(x_{n-1}),\psi(x_n))+qd(\psi(x_{n-1}),\psi(x_n))+rd(\psi(x_n),\psi(x_{n+1}))+s\{d(\psi(x_{n-1}),\psi(x_{n+1}))+d(\psi(x_n),\psi(x_n))\}$$
$$=pd(kx_{n-1},kx_n)+qd(kx_{n-1},kx_n)+rd(kx_n,kx_{n+1})+s\{d(kx_{n-1},kx_{n+1})+d(kx_n,kx_n)\}$$
$$=(pk)d(x_{n-1},x_n)+(qk)d(x_{n-1},x_n)+(rk)d(x_n,x_{n+1})+(sk)\{d(x_{n-1},x_{n+1})+d(x_n,x_n)\}$$
$$=(pk)d(x_{n-1},x_n)+(qk)d(x_{n-1},x_n)+(rk)d(x_n,x_{n+1})+(sk)d(x_{n-1},x_{n+1})$$
$$\le(pk)d(x_{n-1},x_n)+(qk)d(x_{n-1},x_n)+(rk)d(x_n,x_{n+1})+(sk)\{d(x_{n-1},x_n)+d(x_n,x_{n+1})\}\text{(Triangle Inequality)}$$

Thus we have,

$$d(x_n,x_{n+1})\le(pk)d(x_{n-1},x_n)+(qk)d(x_{n-1},x_n)+(rk)d(x_n,x_{n+1})+(sk)\{d(x_{n-1},x_n)+d(x_n,x_{n+1})\}$$
$$\Rightarrow[1-rk-sk]d(x_n,x_{n+1})\le[pk+qk+sk]d(x_{n-1},x_n)$$
$$\Rightarrow d(x_n,x_{n+1})\le\frac{pk+qk+sk}{1-rk-sk}d(x_{n-1},x_n)$$
$$\Rightarrow d(x_n,x_{n+1})\le\frac{p+q+s}{\left(\frac{1}{k}\right)-r-s}d(x_{n-1},x_n)$$
$$\Rightarrow d(x_n,x_{n+1})\le hd(x_{n-1},x_n)$$

Here $h=\frac{p+q+s}{\left(\frac{1}{k}\right)-r-s}<1$ (by condition (B) in the theorem). Repeating this argument, ultimately gives

$$d(x_n,x_{n+1})\le h^n d(x_0,x_1)\qquad(1)$$

When we take the limit as $n\to\infty$ we get $h\to 0$ as $h<1$. Thus it follows from the equation (1) that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy Sequence. Since X is T-Orbitally Complete, there is a point $x\in X$, such that

$$x=\lim_{n\to\infty}T^n x_0=\lim_{n\to\infty}x_n\qquad(2)$$

We shall show that x is a fixed point of T. Using the conditions (B) and (C) in the statement of the theorem and triangle inequality we get,

$d(Tx,Tx_n)$
$\leq pd\left(\psi(x),\psi(x_n)\right)+qd\left(\psi(x),\psi(Tx)\right)+rd\left(\psi(x_n),\psi(Tx_n)\right)+s\left\{d\left(\psi(x),\psi(Tx_n)\right)+d\left(\psi(x_n),\psi(Tx)\right)\right\}$
$=pd\left(\psi(x),\psi(x_n)\right)+qd\left(\psi(x),\psi(Tx)\right)+rd\left(\psi(x_n),\psi(x_{n+1})\right)+s\left\{d\left(\psi(x),\psi(x_{n+1})\right)+d\left(\psi(x_n),\psi(Tx)\right)\right\}$
$=pd\left(kx,kx_n\right)+qd\left(kx,kTx\right)+rd\left(kx_n,kx_{n+1}\right)+s\left\{d\left(kx,kx_{n+1}\right)+d\left(kx_n,kTx\right)\right\}$
$=(pk)d\left(x,x_n\right)+(qk)d\left(x,Tx\right)+(rk)d\left(x_n,x_{n+1}\right)+(sk)\left\{d\left(x,x_{n+1}\right)+d\left(x_n,Tx\right)\right\}$
$\leq(pk)d\left(x,x_n\right)+(qk)\left\{d\left(x,x_{n+1}\right)+d(x_{n+1},Tx)\right\}+(rk)d\left(x_n,x_{n+1}\right)+(sk)\left\{d\left(x,x_{n+1}\right)+d\left(x_n,Tx\right)\right\}$
$=(pk)d\left(x,x_n\right)+\left[qk+sk\right]d(x,x_{n+1})+(qk)d(Tx_n,Tx)+(rk)d(x_n,x_{n+1})+(sk)d\left(x_n,Tx\right)$
$\leq(pk)d\left(x,x_n\right)+\left[qk+sk\right]d(x,x_{n+1})+(qk)d(Tx_n,Tx)+(rk)d(x_n,x_{n+1})+(sk)\left\{d(Tx,Tx_n)+d(Tx_n,x_n)\right\}$
$\leq(pk)d\left(x,x_n\right)+\left[qk+sk\right]d(x,x_{n+1})+\left[qk+sk\right]d(Tx_n,Tx)+\left[rk+sk\right]d(x_n,x_{n+1})$
$\leq\lambda d\left(x,x_n\right)+\lambda d(x,x_{n+1})+\lambda d(Tx_n,Tx)+\lambda d(x_n,x_{n+1})$
$=\lambda\left\{d\left(x,x_n\right)+d(x,x_{n+1})+d(x_n,x_{n+1})\right\}+\lambda d(Tx_n,Tx)$

Consequently,

$$d(Tx,Tx_n)\leq\lambda\left\{d\left(x,x_n\right)+d(x,x_{n+1})+d(x_n,x_{n+1})\right\}+\lambda d(Tx_n,Tx)$$
$$\Rightarrow\left[1-\lambda\right]d(Tx,Tx_n)\leq\lambda\left\{d\left(x,x_n\right)+d(x,x_{n+1})+d(x_n,x_{n+1})\right\}$$
$$\Rightarrow d(Tx,Tx_n)\leq\frac{\lambda}{1-\lambda}\left\{d\left(x,x_n\right)+d(x,x_{n+1})+d(x_n,x_{n+1})\right\}.$$

Taking limit as$n\to\infty$, $\lim_{n\to\infty}d(Tx,Tx_n)\leq\frac{\lambda}{1-\lambda}\lim_{n\to\infty}\{d(x,x_n)+d(x,x_{n+1})+d(x_n,x_{n+1})\}=0$ (by equation (2) and the fact that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy Sequence). Thus $Tx=\lim_{n\to\infty}Tx_n=\lim_{n\to\infty}x_{n+1}=x$. We T has a fixed point $x\in X$.

Uniqueness: Let there is $y\in X$ such that $Ty=y$. Again using the conditions (B) and (C),

$$\begin{gathered}d(x,y)\\=d(Tx,Ty)\\\leq pd(\psi(x),\psi(y))+qd(\psi(x),\psi(Tx))+rd(\psi(y),\psi(Ty))+s\{d(\psi(x),\psi(Ty))+d(\psi(y),\psi(Tx))\}\\=pd(\psi(x),\psi(y))+qd(\psi(x),\psi(x))+rd(\psi(y),\psi(y))+s\{d(\psi(x),\psi(y))+d(\psi(y),\psi(x))\}\\=pd(kx,ky)+s\{d(kx,ky)+d(ky,kx)\}\\=(pk)d(x,y)+2(sk)d(x,y)\\=[pk+2(sk)]d(x,y)\\\leq\lambda d(x,y)\end{gathered}$$

Thus$d(x,y)\leq\lambda d(x,y)$. This implies that $[1-\lambda]d(x,y)\leq 0$. That is $x=y$. Hence the fixed point of T is unique.

Example 4.4: Consider the metric space $X=[0,10]$ with the usual metric d of absolute value. That is $d(x,y)=|x-y|$ for all $x,y\in X$. d satisfies $d(\alpha x,\alpha y)=|\alpha x-\alpha y|=|\alpha||x-y|=|\alpha|d(x,y)$ for all $x,y\in X$. Define $T:X\to X$ by $Tx=\frac{3}{4}x$ for all $x\in X$. Then X is T-Orbitally Complete metric space. Then for$k=\frac{1}{2}$, observe that

1) There exists numbers $p(x,y)=\frac{3}{2}$, $q(x,y)=\frac{1}{10}$, $r(x,y)=\frac{1}{10}$, $s(x,y)=\frac{1}{10}$ such that

$$\underset{x,y\in X=[0,10]}{Sup}\{p+q+r+2s\}=\underset{x,y\in X=[0,10]}{Sup}\left\{\frac{3}{2}+\frac{1}{10}+\frac{1}{10}+2\frac{1}{10}\right\}=\frac{19}{10}=1.9<2=\frac{1}{\left(\frac{1}{2}\right)}=\frac{1}{k}.$$

2) For the function $\psi:X\to X$ defined by $\psi(x)=kx=\frac{1}{2}x$ for all$x\in X$, we have $d(Tx,Ty)=d\left(\frac{3}{4}x,\frac{3}{4}y\right)=\left|\frac{3}{4}x-\frac{3}{4}y\right|=\frac{3}{4}|x-y|$.And

$$pd(\psi(x),\psi(y))+qd(\psi(x),\psi(Tx))+rd(\psi(y),\psi(Ty))+s\{d(\psi(x),\psi(Ty))+d(\psi(y),\psi(Tx))\}$$
$$=\frac{3}{2}d(\psi(x),\psi(y))+\frac{1}{10}d\left(\psi(x),\psi\left(\frac{3}{4}x\right)\right)+\frac{1}{10}d\left(\psi(y),\psi\left(\frac{3}{4}y\right)\right)+\frac{1}{10}\left\{d\left(\psi(x),\psi\left(\frac{3}{4}y\right)\right)+d\left(\psi(y),\psi\left(\frac{3}{4}x\right)\right)\right\}$$
$$=\frac{3}{2}d\left(\frac{1}{2}x,\frac{1}{2}y\right)+\frac{1}{10}d\left(\frac{1}{2}x,\frac{1}{2}\frac{3}{4}x\right)+\frac{1}{10}d\left(\frac{1}{2}y,\frac{1}{2}\frac{3}{4}y\right)+\frac{1}{10}\left\{d\left(\frac{1}{2}x,\frac{1}{2}\frac{3}{4}y\right)+d\left(\frac{1}{2}y,\frac{1}{2}\frac{3}{4}x\right)\right\}$$
$$=\frac{3}{2}\frac{1}{2}d(x,y)+\frac{1}{10}\frac{1}{2}d\left(x,\frac{3}{4}x\right)+\frac{1}{10}\frac{1}{2}d\left(y,\frac{3}{4}y\right)+\frac{1}{10}\frac{1}{2}\left\{d\left(x,\frac{3}{4}y\right)+d\left(y,\frac{3}{4}x\right)\right\}$$
$$=\frac{3}{4}|x-y|+\frac{1}{20}\left(\frac{1}{4}x\right)+\frac{1}{20}\left(\frac{1}{4}y\right)+\frac{1}{20}\left\{\frac{|4x-3y|+|4y-3x|}{4}\right\}$$

Thus clearly $d(Tx,Ty)\le pd(\psi(x),\psi(y))+qd(\psi(x),\psi(Tx))+rd(\psi(y),\psi(Ty))+s\{d(\psi(x),\psi(Ty))+d(\psi(y),\psi(Tx))\}$ for all $x,y\in X$. The conditions (B) and (C) of the theorem 4.4 are satisfied. We see that $x=0$ is the unique fixed point of T in X.

Remark 4.2: In the theorem 4.2, $k=1$gives the Lj. B. Ćirić theorem 3.10. It is also concluded that Kannan fixed point theorem [10] is obtained by replacing a semi-continuous function by the identity function and taking the Kannan constant 1/3 in the theorems 4.1 and 4.2.

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